Exercises for "Approximate shortest paths, distance oracles etc." Summer school of shortest paths (PATH05) — DIKU

1. Show that the expected number of edges selected by the surplus 2 algorithm is $O((mn)^{1/2})$ and that hence the expected running time of the algorithm is $O(n^{3/2}m^{1/2})$.

2. Show that in the surplus 2 algorithm works correctly if instead of selecting each vertex independently with probability $(n/m)^{1/2}$ we select a set of vertices that *dominates* the set of vertices of degree at least $(m/n)^{1/2}$. (A set D dominates a set U if every vertex in U is either in D or has a neighbor in D.) Show that there is always such a dominating set of size $O((mn)^{1/2}\log n)$. How fast can you construct such a set deterministically?

3. Let $G = (V, E)$ be an undirected and unweighted graph. Let A be a set obtained by selecting each vertex, independently, with probability $n^{-1/2}$. Construct a subgraph $G' = (V, E')$ in the following way: Start by adding to G' a tree of shortest paths rooted at each vertex $v \in A$. Next, for each vertex $v \notin A$, if v has a neighbor in A then put in G' an edge connecting it to this neighbor. Otherwise, add all the edges incident on v to G'. Show that G' is an additive 2-spanner of G and that the expected number of vertices in it is $O(n^{3/2})$.

4. Show that the subgraph constructed in the previous exercise is also a multiplicative 3-spanner.

5. Verify the correctness of the spanner construction algorithm of Althöfer et al. How fast can you implement it?

6. Recall the definition of a *cluster* used in the construction of approximate distance oracles: If $w \in A_i - A_{i+1}$, then $C(w) = \{v \in V \mid \delta(w, v) < \delta(A_{i+1}, v)\}$. (Of course $\delta(A, v) = \min_{u\in A} \delta(u, v)$.) Show that if $v \in C(u)$ and v' is on a shortest path from u to v in G, then $v' \in C(u)$. As a consequence, show that it is possible to construct a tree of shortest path of $C(w)$ centered at w. How fast can you construct such a tree of shortest paths?