SHORTEST PATHS: APPLICATIONS, OPTIMIZATION, VARIATIONS, AND SOLVING THE CONSTRAINED SHORTEST PATH PROBLEM

EXERCISES

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1 Applications and Modelling

1.1 Questions from "Network Flows"

Exercises from Ahuja et al.'s book "Network Flows" marked with an asterisk are more highly recommended, and those marked with two asterisks are very strongly recommended.

1. Exercise 4.2 from Ahuja et al.'s book "Network Flows".

2. *Exercise 4.3 from Ahuja et al.'s book "Network Flows". Note that in answering this question, you may assume that all books with same height are stored on same height shelf, i.e. that no splitting of height classes is allowed.

3. *Exercise 4.5 from Ahuja et al.'s book "Network Flows".

4. *Exercise 4.6 from Ahuja et al.'s book "Network Flows".

5. **Exercise 4.7 from Ahuja et al.'s book "Network Flows". Note that in answering this question, you may assume that a concentrator can only be located at one of the nodes. Use c_q to denote the cost of establishing a concentrator at node q, and c_{kq} to denote the cost of homing node k onto a concentrator located at node q.

6. Exercise 4.8 from Ahuja et al.'s book "Network Flows".

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7. **Exercise 4.9 from Ahuja et al.'s book "Network Flows". What is the complexity of the solution approach suggested in this question?

1.2 Systems of Difference Constraints

8. Find a feasible solution or determine that no feasible solution exists for the following system of difference constraints.

$$\begin{aligned} x_1 - x_2 &\leq 1, & x_1 - x_4 &\leq -4, \\ x_2 - x_3 &\leq 2, & x_2 - x_5 &\leq 7, \\ x_2 - x_6 &\leq 5, & x_3 - x_6 &\leq 10, \\ x_4 - x_2 &\leq 2, & x_5 - x_1 &\leq -1, \\ x_5 - x_4 &\leq 3, & x_6 - x_3 &\leq -8 \end{aligned}$$

9. Find a feasible solution or determine that no feasible solution exists for the following system of difference constraints.

$$\begin{aligned} x_1 - x_2 &\leq 4, & x_1 - x_5 &\leq 5, \\ x_2 - x_4 &\leq -6, & x_3 - x_2 &\leq 1, \\ x_4 - x_1 &\leq 3, & x_4 - x_3 &\leq 5, \\ x_4 - x_5 &\leq 10, & x_5 - x_3 &\leq -7, \\ x_5 - x_4 &\leq -8 \end{aligned}$$

10. Can any shortest-path length from the new node, node 0, in the augmented constraint graph, be positive? Explain.

11. Suppose that in addition to a system of difference constraints, we want to handle equality constraints of the form $x_i = x_j + b_{ij}$. Explain how the augmented constraint graph and shortest path approach can be adapted to solve this variety of constraint system.

12. Suppose a system of constraints in variables $x_1, x_2, \ldots, x_n$ contains difference constraints, and single variable bounds of the form $x_i \leq u_i$ or $x_i \geq l_i$. Explain how the augmented constraint graph and shortest path approach can be adapted to solve this variety of constraint system.

2 Optimization and Duality

13. Write down the LP dual of the LP formulation of the shortest path problem, over digraph $G = (V, A)$, with start node s, end node t, and lengths c_{ij} for all $(i, j) \in A$, using dual variables u_i for each node $i \in V$.

    In what follows, you may assume that the shortest path problem is feasible, i.e. that there exists path from s to t in the digraph. Furthermore, for simplicity, you may assume that all nodes in the graph are reachable from node s.

    (a) Explain why it is that we can, without loss of generality, add a constraint $u_s = 0$ to the LP dual.

    (b) Prove that if there is a negative length cycle in the network reachable from s, then the (primal) shortest path LP is unbounded below.

    (c) Explain why it is that if the (primal) shortest path LP is unbounded below, then there must exist a negative length cycle in the network.

    (d) Suppose that there are no negative length cycles in the network. Show that if u_i is taken to be the length of the shortest path from node s to node i for each $i \in V$, then u must be feasible for the LP dual.

    (e) Prove that if the network has no negative length cycle, and the shortest path tree is unique, then the solution to the (primal) shortest path LP is unique, and is the indicator vector for the shortest path from s to t.

14. Write down a (Mixed) Integer Linear Programming formulation of the shortest path problem with renewable node resources. Use the notation $G = (V, A)$ for the digraph, s and t for the path start and end nodes, c for the arc lengths, D for the node resources, Q for the resource limit and $R \subseteq V$ for the nodes at which resources may be renewed. You may assume that any node in R supplies itself, or equivalently that it has zero demand.

3 Variations and Modelling

15. In the case of minefield path planning, the Euclidean length of the path may need to be constrained to be no more than some limit L. Explain how to model this as an additive resource constrained shortest path problem.

16. Suppose that in the case of piecewise linear function approximation, one seeks to minimize the number of points used, subject to constraining the squared error in the approximation to be no worse than some limit L. Show how to model this as an additive resource constrained shortest path problem.

17. Suppose that in the case of piecewise linear function approximation, one seeks to minimize the error in the approximation, subject to constraining the number of points used to be no more than some limit M. Show how to model this as an additive resource constrained shortest path problem.

18. In the case of minefield path planning, the number of turns made along the path may need to be constrained to be no more than some limit M. Show how to model this as an additive resource constrained shortest path problem, in the case that the spatial discretization used is a simple rectangular grid, with arcs only along the sides of each rectangular grid element. [Hint: you may need to introduce new nodes and arcs]. Generalize your model to the case that arcs are allowed along the diagonals of each rectangular grid element.

19. Suggest how you might, in general, formulate the problem of approximating an arbitrary given function $f(x)$ in space between the points $(a, f(a))$ and $(b, f(b))$ by a piecewise linear function, trading off approximation accuracy against the number of linear pieces used, as a shortest path problem in a network. You may assume f has properties that make numerical calculations using f tractable, for example, you may assume f is integrable on the interval $[a, b]$.

20. Given a digraph $G = (V, A)$, path start and end nodes s and t respectively, and a set $T = \{S_1, S_2, \ldots, S_K\}$ of distinct subsets of the nodes, $S_k \subset V$ with $|S_k| \geq 2$ for each $k = 1, \ldots, K$, a *subset disjoint s-t path* P is a path in G from s to t that contains at most one node from any subset. In other words, if we write $V(P)$ to denote the set of nodes in the path P, then P is a subset disjoint path if $|V(P) \cap S_k| \leq 1$ for all $k = 1, \ldots, K$. Given arc lengths c_{ij} for each arc $(i, j) \in A$, the shortest subset disjoint path problem is the problem of finding a subset disjoint s-t path minimizing the sum of c values on the arcs in the path. Show how to model the subset disjoint path problem as a constrained shortest path problem with multiple additive resources.

4 Constrained Shortest Path Problems

4.1 Complexity

In these questions, we are not really concerned with the precise complexity of an algorithm, but are only interested in whether or not a problem (or algorithm) is polynomially solvable, or if, as far as is known, it will require exponential time to solve; in the latter case, we are interested in whether or not it is pseudopolynomially solvable.

21. Prove that the constrained shortest path problems you derived in Questions 17 and 18 are solvable in polynomial time, in the former case in time polynomial in the number of points in the original function, and in the latter case in the number of nodes in the original space discretization network.

22. Show how to solve an additive resource constrained shortest path problem with multiple resources as a shortest path problem in a related graph, i.e. construct this graph. You may assume that at least one of the resources has positive values on every arc, and that all resources are non-negative integers. How many nodes does the graph you constructed have? What does this tell you about the complexity of the additive resource constrained shortest path problem with multiple resources?

23. Use your answer in Question 22 to deduce something about the complexity of the elementary resource constrained shortest path problem, in which the resource used on every arc is assumed to be non-negative.

24. Use your answer in Question 22to discuss the complexity of the constrained shortest path problem you derived in Question 20, in terms of the number of subsets, K.

4.2 Preprocessing

25. Let $G = (V, A)$, where $V = \{1, 2, 3, 4, 5, 6\}$ and A is the set of arcs:

Arc	Resource	Cost
(1,2)	1	1
(1,4)	2	2
(1,3)	1	4
(2,3)	2	2
(2,4)	1	3
(3,4)	3	17
(3,5)	1	13
(3,6)	9	8
(4,3)	2	2
(4,6)	18	4
(5,6)	1	5

The origin is $s = 1$, the destination is $t = 6$, and the resource limit is $R = 19$. Apply the preprocessing procedure.

26. Solve the resource constrained shortest path problem defined in the previous exercise using the Label Setting Algorithm (without preprocessing).